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Research Article

Dyons, Superstrings, and Wormholes: Exact Solutions of the Non-Abelian Dirac-Born-Infeld Action

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We construct dyon solutions on coincident $D4$-branes, obtained by applying T-duality transformations to type I $SO(32)$ superstring theory in 10 dimensions. These solutions, which are exact, are obtained from an action comprising the non-Abelian Dirac-Born-Infeld action and a Wess-Zumino-like action. When one spatial dimension of the $D4$-branes is taken to be vanishingly small, the dyons are analogous to the 't Hooft/Polyakov monopole residing in a 3 + 1-dimensional spacetime, where the component of the Yang-Mills potential transforming as a Lorentz scalar is reinterpreted as a Higgs boson transforming in the adjoint representation of the gauge group. Applying a T-duality transformation to the vanishingly small spatial dimension, we obtain a collection of $D3$-branes, not all of which are coincident. Two of the $D3$-branes, distinct from the others, acquire intrinsic, finite curvature and are connected by a wormhole. The dyons possess electric and magnetic charges whose values on each $D3$-brane are the negative of one another. The gravitational effects, which arise after the T-duality transformation, occur despite the fact that the action of the system does not explicitly include the gravitational interaction. These solutions provide a simple example of the subtle relationship between the Yang-Mills and gravitational interactions, that is, gauge/gravity duality.

1. Introduction

Theoretically appealing but experimentally elusive, the magnetic monopole has captured the interest of the physics community for more than eight decades. The magnetic monopole (an isolated north or south magnetic pole) is conspicuously absent from the Maxwell theory of electromagnetism. In 1931, Dirac showed that the magnetic monopole can be consistently incorporated into the Maxwell theory with virtually no modification to the theory [1]. In addition, Dirac demonstrated that the existence of a single magnetic monopole necessitates not only that electric charge be quantized but also that the electric and magnetic couplings be inversely proportional to each other, the first suggestion of the so-called weak/strong duality. Subsequently, 't Hooft [2] and Alexander Polyakov showed that, within the context of the spontaneously broken Yang-Mills gauge theory $SO(3)$, topological magnetic monopole solutions of finite mass must necessarily exist. Furthermore, these solutions possess an internal structure and also exhibit the same weak/strong duality discovered by Dirac. Consequently, Montonen and Olive conjectured that there exists an exact weak/strong electromagnetic duality for the spontaneously broken $SO(3)$ gauge theory [3]. More recently, this conjecture has become credible within the broader context of $N = 2$ or $N = 4$ Super-Yang-Mills theories. Despite the lack of experimental evidence for the existence of magnetic monopoles, physicists still remain optimistic of their existence. Indeed, Guth proposed the inflationary model of the universe, in part, to explain why magnetic monopoles have escaped discovery [4].

The focus of our investigation is electrically charged magnetic monopole (dyon) solutions within the context of superstring theory. In Section 2, we construct dyon solutions which are exact and closed to first order in the string theory length scale. We, first, begin with a type I $SO(32)$ string theory in ten dimensions, six of the spatial dimensions being compact but arbitrarily large. We, then, apply the group of T-duality transformations to five of the compact spatial dimensions to obtain 16 $D4$-branes, some of which are coincident. The five T-dualized dimensions of each $D4$-brane constitute the internal dimensions of a 4 + 1-dimensional spacetime. Making an appropriate ansatz, we obtain dyon solutions residing on the $D4$-branes. The solutions are based on an action which includes coupling of the $D4$-branes to NS-NS closed strings, the non-Abelian Dirac-Born-Infeld action, and coupling to R-R closed strings, a Wess-Zumino-like action. We next apply a T-duality transformation to the $D4$-branes, resulting in a collection of $D3$-branes, some of which are coincident and two of which are connected by a wormhole. Finally, we interpret the dyon solutions in the context of gauge/gravity duality.

Because of the differences in the literature among the systems of units, sign conventions, and so forth, we present in Appendix A the conventions chosen by us so that direct comparisons can be made between our results and those of other authors.

2. Dyons and Dimensional Reduction of Type I $SO(32)$ Theory

In this section, we construct dyon solutions based on superstring theory. We begin with type I $SO(32)$ superstring theory in ten dimensions [5], six of the spatial dimensions of which are compact. Next, we apply the group of T-duality transformations to five of the compact dimensions letting the size, R, of the dimensions become vanishingly small; that is, $R \to 0$. These five dimensions are the internal dimensions of spacetime. Strictly speaking, spacetime consists of 16 $D4$-branes, bounded by 2^5 orientifold hyperplanes. Each of the $D4$-branes comprises four spatial dimensions, three unbounded and one compact. In what follows, we assume that none of the D-branes are close to the orientifold hypersurfaces. Thus, the theory describing the closed strings in the vicinity of any of the $D4$-branes is type II oriented, rather than type II unoriented. In this particular case, since we have applied the T-duality transformation to an odd number of dimensions, the closed string theory is the type IIa oriented theory. Furthermore, each end of an open string must be attached to a $D4$-brane, which may be the same $D4$-brane or two different $D4$-branes. If we assume that the number of coincident $D4$-branes is n ($2 \leq n \leq 16$), then a $U(n)$ gauge group is associated with the open strings attached to the coincident $D4$-branes. Given these prerequisite conditions, we now construct dyon solutions which reside on these coincident $D4$-branes. These solutions are derived from the D-brane action comprising two parts, the Dirac-Born-Infeld action, S_{DBI}, which couples NS-NS closed strings to the $D4$-brane and the Wess-Zumino-like action, S_{WZ}, which couples R-R closed strings to the $D4$-brane.

2.1. Dyon Solutions on $D4$-Branes. The dyon solutions are obtained from the equations of motion derived from the action, S, which describes the coupling of closed string fields to a general Dp-brane (which in our case is $p = 4$). The action is [6]

$$S = S_{\mathrm{DBI}} + S_{\mathrm{WZ}}, \tag{1}$$

where

$$S_{\mathrm{DBI}} = -\tau_p \int_{\mathscr{M}_{p+1}} \mathrm{STr}\left\{e^{-\Phi}\left[(-1) \cdot \det\left(G_{AB} + B_{AB} + 2\pi\alpha' F_{AB}\right)\right]^{1/2}\right\}, \tag{2}$$

$$S_{\mathrm{WZ}} = \mu_p \int_{\mathscr{M}_{p+1}} \left[\sum_{p'} C_{(p'+1)}\right] \wedge \mathrm{Tr}\, e^{2\pi\alpha' F+B}. \tag{3}$$

Here, τ_p is the physical tension of the Dp-brane, and μ_p is its R-R charge (see Appendix B for a discussion of the relationships among the various string parameters).

The dyon solutions are based on the following ansatz. The dilaton background, Φ, is constant:

$$\Phi = \Phi_0. \tag{4}$$

And background field B vanishes:

$$B_{AB} = 0 \quad (A, B = 0 \cdots 4). \tag{5}$$

The metric G is given by

$$G_{AB} = \widetilde{G}_{AB} I_n, \tag{6}$$

where, for our purposes, $\widetilde{G}_{AB}$ is restricted so that $\widetilde{G}_{00} = -1$ and $\widetilde{G}_{44=1}$.

For $p = 4$, we can reexpress the determinant in (2) as

$$\det\left(G_{AB} + 2\pi\alpha' F_{AB}\right) = \det\left(\widetilde{G}_{AB}\right)\left[I_n + \frac{2\pi\alpha'}{2!} F_{AB} F^{AB} - \frac{\left(2\pi\alpha'\right)^2}{(5-4)!}\left(3^2 \cdot 1^2\right) {}^*(F \wedge F)_E {}^*(F \wedge F)^E\right], \tag{7}$$

where

$${}^*(F \wedge F)_E = \frac{\sqrt{\left|\det\left(\widetilde{G}_{AB}\right)\right|}}{4!} F^{AB} F^{CD} \epsilon_{ABCDE}. \tag{8}$$

See Appendix C, (C.21), for further details.

The term I_n is the n-dimensional identity matrix. The value of n is the dimension of the group $U(n)$ associated with the gauge fields residing on the $D4$-branes. All R-R potentials vanish, except for the one-form potential $C_{(1)}$, which is a constant background field,

$$C_{(1)} = C_4 dx^4, \tag{9}$$

for some constant value C_4. The gauge field, F, is obtained from the gauge potential A ($A = A_E dx^E$), where

$$A_\mu = A_\mu\left(x^i\right), \quad (\mu = 0 \cdots 3; i = 1 \cdots 3), \qquad A_4 = A_4\left(x^i\right). \tag{10}$$

Note that the gauge potentials are static; that is, they do not depend on time, x^0, and also do not depend on the spatial coordinate x^4. The gauge field $F(F = F_{AB} dx^A \wedge dx^B)$, a Lie algebra-valued two-form, is given by

$$F = dA - iA \wedge A. \tag{11}$$

(See Appendix A.) The components of the potentials A_0 and A_4 are constrained in accordance with the condition

$$A_0 \wedge A_4 = 0 \tag{12}$$

so that $F_{04} = 0$. To facilitate its interpretation, we express F_{AB} as a five-dimensional matrix which is explicitly partitioned into electric and magnetic fields which reside in four-dimensional spacetime and an additional component of the magnetic field which resides in the additional space dimension; that is,

$$F_{AB} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 & 0 \\ -E_1 & 0 & B_3 & -B_2 & -\mathscr{D}_1 A_4 \\ -E_2 & -B_3 & 0 & B_1 & -\mathscr{D}_2 A_4 \\ -E_3 & B_2 & -B_1 & 0 & -\mathscr{D}_3 A_4 \\ 0 & \mathscr{D}_1 A_4 & \mathscr{D}_2 A_4 & \mathscr{D}_3 A_4 & 0 \end{pmatrix}. \tag{13}$$

We are seeking dyon solutions. Therefore, with foresight, we make the following assumptions:

$$\begin{aligned} E_i &\equiv F^a_{0i} T_a = F^{(i)}_{0i} T_{(i)}, \\ B_i &\equiv \frac{1}{2}\epsilon_i^{jk} F^a_{jk} T_a = \frac{1}{2}\epsilon_i^{jk} F^{(i)}_{jk} T_{(i)}, \\ \mathscr{D}_i A_4 &\equiv \left(\partial_i A_4 - iA_i \wedge A_4\right)^a T_a = F^a_{4i} T_a = F^{(i)}_{4i} T_{(i)}. \end{aligned} \tag{14}$$

The parenthetical index (i) indicates that there is no summation of that index; however, if an expression contains two indices i without parentheses, then summation of these two indices is implied. Furthermore, each matrix element in (13) includes a generator of $U(n)$; for example, $E_i = E_i^{(i)} T_{(i)}$. Because we are seeking dyon solutions, we may assume without loss of generality that each $T_{(i)}$ is a generator in the fundamental representation of a local $U(1) \times SU(2)$ subgroup of $SU(n)$ (see (44a), (44b), (44c), and (44d)).

The action, (2), can be more straightforwardly interpreted from the perspective of four-dimensional spacetime. Since the action does not depend on the coordinate x^4, we can trivially eliminate x^4 from the action by integrating the x^4 coordinate. As a result of the integration, the tension of the $D4$-brane, τ_4, and the Yang-Mills coupling constant, g_{D4}, are replaced by those of the $D3$-brane, τ_3 and g_{D3} (see (B.3) and (B.5)). Let the size, R_4, of the x^4-dimension become vanishingly small; that is, $R_4 \rightarrow 0$. Then, the field A_4 becomes a Lorentz scalar transforming as the adjoint representation of the gauge group, and (14) gives the covariant derivative of A_4. From the perspective of four spacetime dimensions, A_4 assumes the role of a Higgs boson transforming as the adjoint representation of the gauge group.

Substituting (4)–(6) and (13) into (1) and then integrating the x^4 coordinate, we obtain

$$S_{\mathrm{DBI}} = \int d^{3+1}\xi \mathscr{L}_{\mathrm{DBI}}, \tag{15}$$

where

$$\begin{aligned} &\mathscr{L}_{\mathrm{DBI}} \\ &= -\frac{1}{(2\pi\alpha')^2 g_{D3}^2} \mathrm{STr}\left[\left\{\left|\det\left(G_{AB} + 2\pi\alpha' F_{AB}\right)\right|^{1/2}\right\}\right] \\ &= -\frac{\sqrt{\left|\det\left(\widetilde{G}_{ij}\right)\right|}}{(2\pi\alpha')^2 g_{D3}^2} \mathrm{STr}\left\{\mathscr{L}'\right\}. \end{aligned} \tag{16}$$

The function $\mathscr{L}'$ is defined as

$$\begin{aligned} \mathscr{L}' = \Big\{ & I_n - \left(2\pi\alpha'\right)^2 \left(E \cdot E - B \cdot B - \mathscr{D}A_4 \cdot \mathscr{D}A_4\right) \\ & + \left(2\pi\alpha'\right)^4 \left(B \cdot \mathscr{D}A_4\right)^2 - \left(2\pi\alpha'\right)^4 (E \cdot B)^2 \\ & - \left(2\pi\alpha'\right)^4 \left(E \times \mathscr{D}A_4\right) \cdot \left(E \times \mathscr{D}A_4\right)\Big\}^{1/2}. \end{aligned} \tag{17}$$

We have used the fact that $\sqrt{|\det(\widetilde{G}_{ij})|} = \sqrt{|\det(\widetilde{G}_{AB})|}$. In (16), the ordering of the generators of the algebra, T_a, corresponds with the order of the fields as they appear in the equation; for example,

$$\begin{aligned} \left\{B \cdot \mathscr{D}A_4\right\}^2 &= \left\{B \cdot \mathscr{D}A_4\right\}\left\{B \cdot \mathscr{D}A_4\right\} \\ &= \left\{\widetilde{G}^{ij} B_i^{(i)} T_{(i)} \left(\mathscr{D}_j A_4\right)^{(j)} T_{(j)}\right\} \\ &\quad \cdot \left\{\widetilde{G}^{ij} B_i^{(i)} T_{(i)} \left(\mathscr{D}_j A_4\right)^{(j)} T_{(j)}\right\}. \end{aligned} \tag{18}$$

Note that "STr" indicates that the trace is calculated symmetrically; that is, the trace is symmetrized with respect to all gauge indices [6, 7]. The implication is that the evaluation of the trace requires that after the expansion of (16) in powers of the field strengths, all orderings of the field strengths are included with equal weight; that is, products of T_a are replaced by their symmetrized sum, before the trace is evaluated. This is discussed in detail in [6, 7].

In (16), the dot product and cross product of two 3-vectors, for example, E and $\mathscr{D}A_4$, are defined as $E \cdot \mathscr{D}A_4 = \widetilde{G}^{ij} E_i \mathscr{D}_j A_4$ and $(E \times \mathscr{D}A_4)_i = \varepsilon_i^{jk} E_j \mathscr{D}_k A_4$.

In obtaining (16), we have reexpressed the dilaton Φ, on a $D4$-brane, in terms of the dilaton Φ', on a $D3$-brane, both of which are related by a T-duality transformation in the x^4-dimension. Specifically, Φ and Φ' are related by $e^{\Phi'} = \alpha'^{1/2} e^{\Phi} / R_4$. The constant dilaton background ϕ_0 has been incorporated into the physical tension τ_p (see Appendix B).

Substituting (5), (9), (6), and (13) into (3), we obtain

$$S_{\mathrm{WZ}} = \frac{\mu_4}{2!} \int_{\mathscr{M}_5} C_{(1)} \wedge \mathrm{Tr}\left\{2\pi\alpha' F \wedge 2\pi\alpha' F\right\}. \tag{19}$$

Integrating the x^4 coordinate in (19), we obtain

$$S_{\mathrm{WZ}} = \int d^{3+1}\xi \mathscr{L}_{\mathrm{WZ}}, \tag{20}$$

where

$$\begin{aligned}\mathscr{L}_{\mathrm{WZ}} &= \frac{\theta}{4\pi^2}\operatorname{Tr}\{F\wedge F\} = \sqrt{\left|\det\left(\widetilde{G}_{ij}\right)\right|}\frac{\theta}{4\pi^2}\operatorname{Tr}\{E\cdot B\} \\ &= \sqrt{\left|\det\left(\widetilde{G}_{ij}\right)\right|}\frac{\theta}{4\pi^2}E^{i(i)}B_i^{(i)}\operatorname{STr}\{T_{(i)}T_{(i)}\} \\ &= \sqrt{\left|\det\left(\widetilde{G}_{ij}\right)\right|}\frac{\theta}{8\pi^2}E^{i(i)}B_i^{(i)},\end{aligned} \tag{21}$$

where $E^{i(i)} = F^{0i(i)}T_{(i)}$. Here,

$$\theta \equiv \frac{C_4}{2!}\frac{2\pi R_4}{\alpha'^{1/2}}. \tag{22}$$

In obtaining (21), we have explicitly evaluated μ_4 using (B.4). Equation (21) is associated with the Witten effect. Witten has demonstrated that adding term (21) to the Lagrangian of Yang-Mills theory does not alter the classical equations of motion but does alter the electric charge quantization condition in the magnetic monopole sector of the theory [5, 8, 9]. In summary, the action, S, for the $D4$-brane is given by

$$S = \int d^{3+1}\xi \mathscr{L}, \tag{23}$$

where

$$\mathscr{L} = \mathscr{L}_{\mathrm{DBI}} + \mathscr{L}_{\mathrm{WZ}}. \tag{24}$$

The equations of motion which are obtained from (23) are

$$\mathscr{D}_\mu P^{\mu\nu} = 0, \tag{25}$$

where

$$P^a_{\mu\nu} = \frac{\partial\mathscr{L}}{\partial F^{\mu\nu a}}. \tag{26}$$

In addition, the fields $F^a_{\mu\nu}$ satisfy the Bianchi identity

$$\mathscr{D}_{[\alpha}F^a_{\beta\gamma]} = 0. \tag{27}$$

To facilitate the ensuing analysis, we transform the Lagrangian density, $\mathscr{L}$, to the Hamiltonian density, $\mathscr{H}$, using the Legendre transformation

$$\mathscr{H} = \operatorname{STr}\{P_0\cdot E - \mathscr{L}\}, \tag{28}$$

where

$$\begin{aligned}P_{0i}^{(i)} &\equiv \frac{\partial\mathscr{L}}{\partial E^{i(i)}} \\ &= -\frac{\sqrt{\left|\det\left(\widetilde{G}_{ij}\right)\right|}}{\left(2\pi\alpha'\right)^2 g_{D3}^2}\operatorname{STr}\left\{\frac{X_i^{(i)}}{\mathscr{L}'}\right\} \\ &\quad+ \sqrt{\left|\det\left(\widetilde{G}_{ij}\right)\right|}\frac{\theta}{4\pi^2}B_i^{(i)}\operatorname{STr}\{T_{(i)}T_{(i)}\},\end{aligned} \tag{29}$$

where

$$\begin{aligned}X_i^{(i)} &= E_iT_{(i)} + (E\cdot B)\,T_{(i)}B_i \\ &\quad+ \left(\mathscr{D}A_4\times[E\times\mathscr{D}A_4]\right)_i T_{(i)}.\end{aligned} \tag{30}$$

After performing detailed calculations, we obtain

$$\mathscr{H} = \frac{\sqrt{\left|\det\left(\widetilde{G}_{ij}\right)\right|}}{\left(2\pi\alpha'\right)^2 g_{D3}^2}\operatorname{STr}\left\{\mathscr{H}'\right\}, \tag{31}$$

where

$$\begin{aligned}\mathscr{H}' = \Bigg\{ & I_n + \left(2\pi\alpha'\right)^2\left(P_0\cdot P_0 + B\cdot B + \mathscr{D}A_4\cdot\mathscr{D}A_4\right) \\ & + \left(2\pi\alpha'\right)^4\Bigg(\left[B\cdot\mathscr{D}A_4\right]^2 + \left[P_0\cdot\mathscr{D}A_4\right]^2 \\ & + \frac{\left[P_0\times B\right]^2 + \left[\mathscr{D}A_4\times\left(P_0\times B\right)\right]^2}{\left(2\pi\alpha'\right)^{-2}I_n + \mathscr{D}A_4\cdot\mathscr{D}A_4}\Bigg)\Bigg\}^{1/2}.\end{aligned} \tag{32}$$

The electric field $E_i^{(i)}$ can be expressed as a function of $P_{0i}^{(i)}$:

$$E_i^{(i)} = \frac{\partial\mathscr{H}}{\partial P_{0i}^{(i)}} = \frac{\sqrt{\left|\det\left(\widetilde{G}_{ij}\right)\right|}}{\left(2\pi\alpha'\right)^2 g_{D3}^2}\operatorname{STr}\left\{\frac{Y_i^{(i)}}{\mathscr{H}'}\right\}. \tag{33}$$

The term $Y_i^{(i)}$ is given by

$$\begin{aligned}Y_i^{(i)} = {} & \left(2\pi\alpha'\right)^2 P_{0i}T_{(i)} + \left(2\pi\alpha'\right)^4 \\ & \cdot\Bigg[\mathscr{D}_iA_4T_{(i)}\left(P_0\cdot\mathscr{D}A_4\right) \\ & + \left(B\times\left(P_0\times B\right)\right)_i T_{(i)} \\ & - \frac{\left(B\times\mathscr{D}A_4\right)_i T_{(i)}\left(P_0\cdot\left(B\times\mathscr{D}A_4\right)\right)}{\left(2\pi\alpha'\right)^{-2}I_n + \mathscr{D}A_4\cdot\mathscr{D}A_4}\Bigg].\end{aligned} \tag{34}$$

We seek dyon solutions which are BPS states, that is, whose energy $\mathscr{E}$ ($\mathscr{E} = \int d^3\xi\mathscr{H}$) is a local minimum. First, we reexpress $\mathscr{H}$

$$\mathcal{H} = \frac{\sqrt{\left|\det\left(\widetilde{G}_{ij}\right)\right|}}{\left(2\pi\alpha'\right)^2 g_{D3}^2}\mathrm{STr}\left[\left\{\left[I_n + \left(2\pi\alpha'\right)^2\left(\cos\phi P_0\cdot\mathscr{D}A_4 + \sin\phi B\cdot\mathscr{D}A_4\right)\right]^2 + \left(2\pi\alpha'\right)^2\left[\left(\sin\phi P_0\cdot\mathscr{D}A_4 - \cos\phi B\cdot\mathscr{D}A_4\right)^2 + \left(P_0 - \cos\phi\mathscr{D}A_4\right)^2 + \left(B - \sin\phi\mathscr{D}A_4\right)^2\right] + \left(2\pi\alpha'\right)^4\frac{\left[P_0\times B\right]^2 + \left[\mathscr{D}A_4\times\left(P_0\times B\right)\right]^2}{\left(2\pi\alpha'\right)^{-2}I_n + \mathscr{D}A_4\cdot\mathscr{D}A_4}\right\}^{1/2}\right]. \tag{35}$$

The mixing angle, ψ, between the electric and magnetic fields of the dyon is defined as

$$\tan\psi = \frac{g_m}{g_e}. \tag{36}$$

The quantities g_m and g_e are the electric and magnetic charges, respectively, of the dyon. The energy, $\mathscr{E}$, is minimized by constraining the dyon solutions to satisfy

$$P_0 = \cos\psi\mathscr{D}A_4, \tag{37a}$$

$$B = \sin\psi\mathscr{D}A_4. \tag{37b}$$

In (35), the second and third squared terms are zero as a consequence of the constraint. Since $P_0 \propto B$, the fourth squared term is also zero by virtue of

$$\left(P_0\times B\right)^k = \frac{P_{0i}^{(i)}B_{0j}^{(j)} - P_{0j}^{(j)}B_{0i}^{(i)}}{2!}\left(T_{(i)}T_{(j)}\varepsilon_{ij}^k - if_{ij}^{(k)}T_{(k)}\right). \tag{38}$$

Thus, $\mathscr{H}$ simplifies so that the energy is

$$\mathscr{E} = \frac{1}{\left(2\pi\alpha'\right)^2 g_{D3}^2}\int d^3\xi\sqrt{\left|\det\left(\widetilde{G}_{ij}\right)\right|}\,\mathrm{Tr}\left[I_n + \left(2\pi\alpha'\right)^2\left(\cos\phi P_0\cdot\mathscr{D}A_4 + \sin\phi B\cdot\mathscr{D}A_4\right)\right]. \tag{39}$$

Substituting (37a) and (37b) into (32) through (34) and using (39), we find

$$E = P_0. \tag{40}$$

In (39), there are two terms which contribute to the mass of the system. The first term within the trace, that is, I_n, corresponds to the volume of each coincident $D4$-brane (or $D3$-brane), which is infinite because the D-branes are not compact. The second term, by virtue of the equations of motion, (26), and the Bianchi identity, (27), can be expressed as a divergence and is therefore a topological invariant. The second term corresponds to the mass of the dyon and is proportional to $\sqrt{g_e^2 + g_m^2}$ as discussed below.

The solutions to (25) and (27) can be straightforwardly obtained from the dyon solutions derived in [10]. Adapting the notation of [10] to the notation used here, we express the vector potential A, (10), in the form (in accordance with our conventions, the Yang-Mills coupling constant appears explicitly in the Lagrangian (A.1). In [8, 10], the coupling constant has been incorporated into the Yang-Mills fields. Thus, to compare results here with those in the references, the fields A and related fields should be divided by g_{D3})

$$\begin{aligned} A &= A_\mu dx^\mu + A_4 dx^4 \\ &= \cos\psi S(r)\, g_{D3}v\alpha_1 T_r dt \\ &\quad + W(r)\left[T_\theta\sin(\theta)\, n d\phi - T_\phi d\theta\right] \\ &\quad + g_{D3}v\left[\alpha_2 T_\perp + Q(r)\,\alpha_1 T_r\right]dx^4, \end{aligned} \tag{41}$$

where v is an arbitrary constant. For the Lie group $SU(n)$

$$\alpha_1 = \sqrt{\frac{n}{2(n-1)}}, \tag{42a}$$

$$\alpha_2 = -\sqrt{\frac{n-2}{2(n-1)}}. \tag{42b}$$

Here, T_i $(i = r,\theta,\phi)$ constitute a representation of the $SU(2)$ subalgebra and $T_\perp$ commutes with each T_i. The quantities r,θ,ϕ are the spherical polar coordinates in three dimensions. The elements T_r, T_θ, T_ϕ are related to T_x, T_y, T_z:

$$T_r = T_x\sin\theta\cos n_m\phi + T_y\sin\theta\sin n_m\phi + T_z\cos\theta, \tag{43a}$$

$$T_\theta = T_x\cos\theta\cos n_m\phi + T_y\cos\theta\sin n_m\phi - T_z\sin\theta, \tag{43b}$$

$$T_\phi = -T_x\sin n_m\phi + T_y\cos n_m\phi. \tag{43c}$$

For $SU(n)$, the n-dimensional matrices T_x, T_y, T_z, and $T_\perp$ are given by

$$T_x = \frac{1}{2}\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \tag{44a}$$

$$T_y = \frac{1}{2}\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & -i \\ 0 & \cdots & 0 & i & 0 \end{pmatrix}, \quad (44b)$$

$$T_z = \frac{1}{2}\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix}, \quad (44c)$$

$$T_\perp = \frac{1}{2\sqrt{n(n-2)}}\begin{pmatrix} -2 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \cdots & -2 & 0 & 0 \\ 0 & \cdots & 0 & n-2 & 0 \\ 0 & \cdots & 0 & 0 & n-2 \end{pmatrix}. \quad (44d)$$

T_x, T_y, T_z, and $T_\perp$ are suitable linear combinations of specific elements of the Cartan subalgebra of $SU(n)$ (see [10] for details). The value of the integer n_m in (43a), (43b), and (43c) is the integer multiple of the fundamental unit of dyon's magnetic charge.

These results differ from those of [10]. For a direct comparison, first replace the azimuthal angle, ϕ, in [10] with ϕ' and extend the domain from $[0, 2\pi]$ to $[0, 2\pi n_m]$; that is, $\phi' \in [0, 2\pi n_m]$. Now, perform the change of variables $\phi' = n_m\phi$ to the dyon solutions of [10] to obtain those given in (41). In addition, apply the same change of variables to the metric in [10] to obtain the metric $\widetilde{G}_{ij}$:

$$\widetilde{G}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 n_m^2 \sin^2\theta \end{pmatrix}. \quad (45)$$

Here, $r \in [0, \infty]$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi]$. This generalizes the results of [10] which only applies to dyons with one unit of magnetic charge; that is, $g_m = 1/g_{D3}$.

The solutions $W(r)$, $Q(r)$, and $S(r)$ are obtained as in [10]

$$W(r) = w(x) = 1 - \frac{x}{\sinh x}, \quad (46a)$$

$$Q(r) = q(x) = \coth x - \frac{1}{x}, \quad (46b)$$

$$S(r) = s(x) = q(x) = \coth x - \frac{1}{x}, \quad (46c)$$

where the dimensionless variable x is related to the radial coordinate r:

$$x = \sin\psi g_{D3} v \alpha_1 r. \quad (47)$$

The field tensor F_{AB} of a dyon with electric charge g_e and magnetic charge g_m,

$$g_m = \frac{n_m}{g_{D3}}, \quad (48)$$

can now be obtained from (41). Specifically,

$$\begin{aligned} F_{tr} &= \frac{g_e}{g} S'(r)\, g_{D3} v \alpha_1 T_r, \\ F_{t\theta} &= [1 - W(r)]\, \frac{g_e}{g} S(r)\, g_{D3} v\alpha_1 T_\theta, \\ F_{t\phi} &= [1 - W(r)]\, \frac{g_e}{g} S(r)\, g_{D3} v\alpha_1 n_m \sin\theta T_\phi, \\ F_{r\theta} &= -W'(r)\, T_\phi, \\ F_{\phi r} &= -W'(r)\, n_m \sin\theta T_\theta, \\ F_{\theta\phi} &= -W(r)\,(2 - W(r))\, n_m \sin\theta T_r, \end{aligned} \quad (49)$$

$$\begin{aligned} D_r A_4 &= Q'(r)\, g_{D3} v\alpha_1 T_r, \\ D_\theta A_4 &= [1 - W(r)]\, Q(r)\, g_{D3} v\alpha_1 T_\theta, \\ D_\phi A_4 &= [1 - W(r)]\, Q(r)\, g_{D3} v\alpha_1 n_m \sin\theta T_\phi. \end{aligned} \quad (50)$$

We now show that gauge invariance of action, (23), implies $SL(2, Z)$ invariance. Consider $U(1)$ gauge transformations which are constant at infinity and are also rotations about the axis $\widehat{A}_4 = A_4/|A_4|$, specifically the gauge transformations [8]

$$\delta A^a_\mu = \frac{1}{g_{D3} v\alpha_1} \left(\mathscr{D}_\mu A_4\right)^a. \quad (51)$$

Action (23) is invariant under these gauge transformations. According to the Noether method, the generator of these gauge transformations, $\mathscr{N}$, is given by

$$\mathscr{N} = \frac{\partial \mathscr{L}}{\partial \partial_0 A^a_\mu} \delta A^a_\mu. \quad (52)$$

Substituting the Lagrangian density (24) into (52), we obtain

$$\mathscr{N} = \frac{\mathscr{G}_e}{g_{D3}} + \frac{g_{D3}^2 \theta \mathscr{G}_m}{8\pi^2}, \quad (53)$$

where

$$\mathscr{G}_m = \frac{1}{v\alpha_1} \int d^3\xi \operatorname{STr}\{\mathscr{D}A_4 \cdot B\}, \quad (54a)$$

$$\mathscr{G}_e = \frac{1}{g_{D3}^2 v\alpha_1} \int d^3\xi \operatorname{Tr}\{\mathscr{D}A_4 \cdot P_0\} \quad (54b)$$

are the magnetic and electric charge operators. Since rotations of 2π about the axis $\widehat{A}_4$ must yield the identity for physical states, that is,

$$e^{2\pi i \mathscr{N}} = 1, \quad (55)$$

applying the $U(1)$ transformation on the left side of (55) to states in the adjoint representation of $SU(n)$, we find that the eigenstates of $\mathcal{N}$ are quantized with eigenvalue

$$\mathcal{N} = \alpha_1 \eta, \tag{56}$$

where η is an arbitrary integer. Substituting (56) into (53), we obtain

$$g_e = \alpha_1 \left[\eta g_{D3} - \frac{\theta'}{2\pi} n_m g_{D3} \right], \tag{57}$$

where we have defined θ' by

$$\theta \equiv \alpha_1 \theta', \tag{58}$$

and used the fact that

$$g_m g_{D3} = n_m 4\pi. \tag{59}$$

Taking $\theta' = 0$ in (57), we obtain the quantization condition for the electric charge

$$g_e = \eta \alpha_1 g_{D3}. \tag{60}$$

The electromagnetic contribution to the mass (rest energy) of the dyon, m_{em}, can be obtained by substituting (54a) and (54b) into (39) and integrating the second term within the trace to obtain

$$m_{\text{em}} = v \alpha_1 \sqrt{g_e^2 + g_m^2}. \tag{61}$$

We can now make $SL(2, Z)$ symmetry explicit. We first define

$$\tau = \frac{\theta'}{2\pi} + \frac{4\pi}{g_{D3}^2} i. \tag{62}$$

If $\theta' = 0$, then the weak/strong duality condition $g_{D3} \to g_m = (4\pi)/g_{D3}$ is equivalent to

$$\tau \longrightarrow -\frac{1}{\tau}. \tag{63}$$

In (57), the transformation $\theta' \to \theta' + 2\pi$ results in identical physical systems with only states being relabeled. The transformation is equivalent to

$$\tau \longrightarrow \tau + 1. \tag{64}$$

Transformations (63) and (64) generate the group $SL(2, Z)$. See [8, 9] for further details.

Note that in (54a) and (54b) $\mathcal{G}_e$ is, strictly speaking, not the electric charge operator because P_0 is not the electric field but rather is its conjugate; however, according to (33) and (34), if $\mathcal{D} A_4$ and B become vanishingly small for asymptotically large values of the radial coordinate, then P_0 approaches E. Thus, in the asymptotic limit $\mathcal{G}_e$ is the electric charge operator. This distinguishing feature is a direct consequence of the fact that our analysis is based on the Born-Infeld action rather than the Yang-Mills-Higgs action. In our case, this point is inconsequential since $P_0 = E$, exactly.

2.2. Dyon Solutions on D3-Branes. As emphasized previously, the dyon solutions derived in Section 2.1, when interpreted from 3 + 1 spacetime dimensions, that is, the compactified theory in which $R_4 \to 0$, are the 't Hooft/Polyakov magnetic monopole or dyon, with the potential A_4 being a Higgs boson transforming in the adjoint representation of the gauge group $U(n)$. Here, our purpose is to reinterpret these dyon solutions in which $R_4 \to 0$ from the equivalent T-dual theory. In the T-dual theory, the radius R_4 is replaced by R_4' ($R_4' = \alpha'/R_4$) so that the radius of the x^4-dimension $R_4' \to \infty$. In addition, the potential A_4 is reinterpreted as the x^4-coordinates of the n $D3$-branes embedded in 4 + 1 dimensional spacetime. These coordinates can be directly obtained by diagonalizing A_4, (41), using a local gauge transformation which rotates T_r into T_z. The n x^4-coordinates are the diagonal elements of the matrix; that is (we are assuming that after the T-duality transformation the $D3$-branes are far from any orientifold hyperplanes. This can always be accomplished by adding the to A_4 component of the gauge potential a constant $U(1)$ gauge transformation $\theta_0 T^0$, θ_0 being a suitable constant (see Appendix A)),

$$\begin{aligned} A_4 &\longrightarrow 2\pi\alpha' g_{D3} v \left[\alpha_2 T_\perp + Q(r) \alpha_1 T_z \right] \\ &= 2\pi\alpha' g_{D3} v \begin{pmatrix} u_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \cdots & u_1 & 0 & 0 \\ 0 & \cdots & 0 & u_2 + \tilde{u}(r) & 0 \\ 0 & \cdots & 0 & 0 & u_2 - \tilde{u}(r) \end{pmatrix}, \end{aligned} \tag{65}$$

where

$$u_1 = -\frac{\alpha_2}{\sqrt{n(n-2)}}, \tag{66a}$$

$$u_2 = \frac{\alpha_2}{2} \sqrt{\frac{n-2}{n}}, \tag{66b}$$

$$\tilde{u}(r) = \frac{\alpha_1}{2} Q(r). \tag{66c}$$

Of the n $D3$-branes $n-2$ of the $D3$-branes, denoted by $D3_{n-2}$, are coincident. The x^4-coordinate of each is the constant value $2\pi\alpha' g_{D3} v u_1$. For the remaining two $D3$-branes, denoted by $D3_1$ and $D3_2$, the x^4-coordinate of each is a function of the radial coordinate r. Specifically, $x^4 = 2\pi\alpha' g_{D3} v(u_2 - \tilde{u}(r))$ for $D3_1$ and $x^4 = 2\pi\alpha' g_{D3} v(u_2 + \tilde{u}(r))$ for $D3_2$, and as a consequence these two $D3$-branes have nonvanishing intrinsic curvature. This occurs despite the fact that before the application of the T-duality transformation no gravitational interaction is explicitly present. We now introduce the length scale L_{D3} which is the separation between $D3_1$ and $D3_2$, in the asymptotic limit as the radial coordinate $r \to \infty$. It is related to previously defined parameters by

$$L_{D3} = 2\pi\alpha' g_{D3} v \alpha_1. \tag{67}$$

Another relevant length scale is the size of the dyon, that is, the region of space where all components of the Yang-Mills field, F_{AB}, are nonvanishing. According to (49), (46a), (46b), (46c), and (47), only the radial components of the electric and magnetic fields are long range, with the remaining components of the fields vanishing exponentially for $x \gg 1$. Thus, additional structure of the dyon becomes apparent whenever $x \lesssim 1$ or equivalently whenever $r \lesssim 1/(\sin\psi g_{D3} v\alpha_1)$. We can therefore define the size of dyon L_d, as measured from asymptotically flat space, that is, $r \to \infty$, to be

$$L_d = \frac{1}{\sin\psi g_{D3} v\alpha_1} = \frac{2\pi\alpha'}{\sin\psi L_{D3}}. \tag{68}$$

In Figure 1, we show, for the gauge group $SU(5)$, embedding plots of the 5 $D3$-branes as a function of the dimensionless radial coordinate, x ($x = r/L_d$). As $x \to \infty$, the x^4-coordinate of $D3_2$ approaches that of the (5–2) coincident $D3$-branes, $D3_{5-2}$, in effect, joining them by a wormhole in an asymptotically flat region of space. At $x = 0$, $D3_2$ is joined to $D3_1$ by another wormhole. (See [11] for a recent discussion of thin shell wormholes exhibiting cylinder symmetry.) As we will show, in general, the intrinsic curvature of the two surfaces in the neighborhood of $x = 0$ is relatively large but, nonetheless, finite. Although these features described in Figure 1 apply to the particular gauge group $SU(5)$, they apply to all $SU(n)$, $n \geq 2$. (For $n = 2$ there are no coincident $D3$-branes.)

We now consider in detail the two $D3$-branes, D_1 and D_2, whose x^4-coordinates are radial dependent. The T-duality transformation on the x^4-dimension pulls back the metric onto D_1 and D_2, inducing the metric, G'_{ij},

$$G'_{ij} = \widetilde{G}_{ij} I_n + \left(\mathscr{D}_i A_4\right)^{(i)} T_{(i)} \mathscr{D}_j A_4^{(j)} T_{(j)} \delta_{(i)(j)} \widetilde{G}_{44} I_n. \tag{69}$$

Expanding the right-hand side of (69), we obtain

$$G'_{ij} = \begin{pmatrix} \widetilde{G}_{ij} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \cdots & \widetilde{G}_{ij} & 0 & 0 \\ 0 & \cdots & 0 & \widetilde{G}_{ij} + \widetilde{A}_{ij} & 0 \\ 0 & \cdots & 0 & 0 & \widetilde{G}_{ij} + \widetilde{A}_{ij} \end{pmatrix}, \tag{70}$$

where

$$\widetilde{A}_{ij} = \left(\frac{g_{D3} L\alpha_1}{2}\right)^2 \times \begin{pmatrix} \left[Q'(r)\right]^2 & 0 & 0 \\ 0 & \widetilde{W}^2(r) & 0 \\ 0 & 0 & \widetilde{W}^2(r)\, n_m^2 \sin^2\theta \end{pmatrix}. \tag{71}$$

Here,

$$\widetilde{W}(r) = \left[1 - W(r)\right] Q(r), \tag{72}$$

and $\widetilde{G}_{ij}$ is given by (45). In obtaining (70), we have used (50) and the fact that the matrices T_r, T_θ, and T_ϕ, (43a), (43b), and (43c), satisfy the relationship

$$T_r^2 = T_\theta^2 = T_\phi^2 = \left(\frac{1}{2}\right)^2 \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}. \tag{73}$$

Each of the diagonal entries in the matrix G'_{ij} corresponds to the metric on one of the n $D3$-branes obtained from the T-duality transformation. In the case of the first $n-2$ entries, corresponding to the $D3$-branes $D3_{n-2}$, the metric is flat. In the case of the last two entries corresponding to the $D3$-branes, $D3_1$ and $D3_2$, their geometries are identical and intrinsically curved. The only feature which distinguishes these two $D3$-branes is that the function $Q(r)$ defining the x^4-coordinate for $D3$-brane D_2 is replaced by $-Q(r)$ for $D3_1$, as evidenced in (65) and (66a) and (66b) and (66c) and also in Figure 1. As a consequence, the electric and magnetic charges of the dyon on D_2 are minus the values on D_1. The electric and magnetic field lines enter the wormhole from one $D3$-brane and exit from the other. Figure 2 is an embedding diagram showing the $D3$-branes D_1 and D_2 in the neighborhood of the radial coordinate $r = 0$. As there is no event horizon surrounding $r = 0$, the two $D3$-branes are joined by a wormhole at $r = 0$.

Of particular interest is the intrinsic scalar curvature of $D3_1$ and $D3_2$, in the neighborhood of $r = 0$. The scalar curvature can be calculated from the metric G'_{ij}, (70), and its value $R(0)$ at $r = 0$ is

$$R(0) = 216 \sin\psi \frac{L_{D3}^6 \sin^3\psi}{\left[L_{D3}^4 \sin^2\psi + \left(12\pi\alpha'\right)^2\right]^2}. \tag{74}$$

For a given value of $\sin\psi$, $R(0)$ assumes its maximum value $\widetilde{R}(0)$:

$$\widetilde{R}(0) = \frac{27\sqrt{3}}{8} \frac{\sin\psi}{\pi\alpha'}, \tag{75}$$

when $L_{D3} = \widetilde{L}_{D3}$, where

$$\widetilde{L}_{D3} = \left(\frac{12\sqrt{3}\pi\alpha'}{\sin\psi}\right)^{1/2}. \tag{76}$$

For either $L_{D3} \to 0$ or $L_{D3} \to \infty$, the scalar curvature $R(0) \to 0$; that is, the geometry of D_1 and D_2 becomes flat, everywhere. The expression for the scalar curvature $R(r)$ is a complicated function of r and not amenable to straightforward interpretation and, therefore, will not be given. In Figure 3, we show a plot of the scalar curvature as a function of the radial coordinate. In this example, $L_{D3} = \sqrt{\alpha'}$, and the dyon has only one unit of magnetic charge so that $\sin\psi = 1$. Near $r = 0$, the scalar curvature is positive and

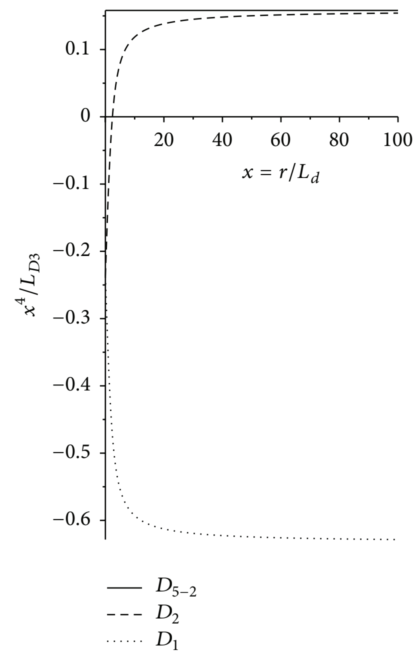

Figure 1: Embedding Functions of the 5 $D3$-branes for the gauge group $SU(5)$. The scaled coordinate x^4/L_{D3} is plotted as a function of the scaled radial coordinate, r/L_d, for the 5 $D3$-branes. The radial coordinate, r, is scaled by the size of the dyon, L_d, and the x^4 coordinate is scaled by the separation between the two $D3$-branes $D3_1$ and $D3_1$ in the asymptotic limit of large r.

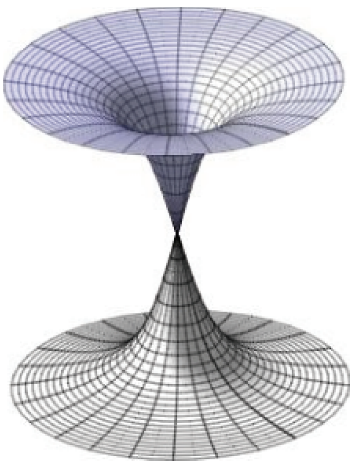

Figure 2: Wormhole. Shown is the embedding diagram for the two $D3$-branes, $D3_1$ and $D3_2$, with azimuthal angle suppressed. The domain of the radial coordinate, r, is $0 \leq r \leq 15L_d$, and the range of the embedding coordinate x^4 is $-.45L_{D3} \leq x^4 \leq +.45L_{D3}$.

finite. As r increases, the scalar curvature becomes slightly negative and asymptotically approaches zero as $r \rightarrow \infty$. These features of the scalar curvature described for this specific example also apply in general.

Consider dyon solutions for which $L_{D3} \approx \sqrt{\alpha'}$ or less. The F-strings connecting $D3_1$ and $D3_2$ would be in their ground state, a BPS state. In addition, assume that $g_{D3} \ll 1$; then as $r \rightarrow 0$ from an asymptotically flat region of space, within

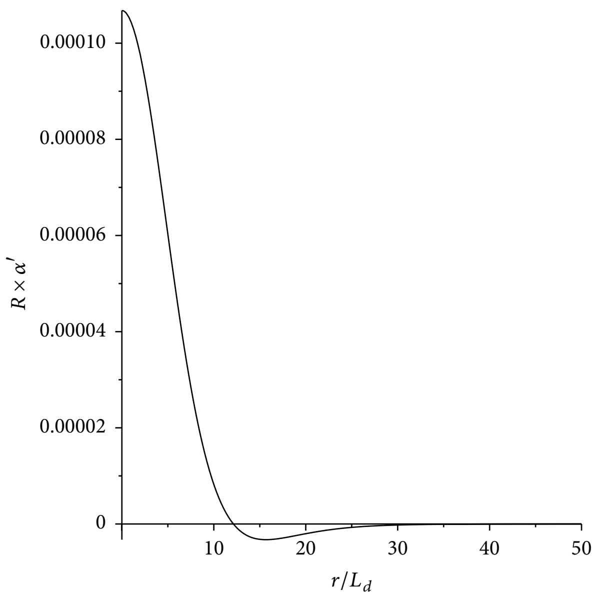

Figure 3: Scalar curvature $R(r)$. For the case $L_{D3} = \sqrt{\alpha'}$ and $\sin\psi = 1$, the dimensionless scalar curvature, $R \times \alpha'$, is plotted as a function of the scaled radial coordinate, r/L_d.

either $D3_1$ or $D3_2$, the string length scale will be reached before the gravitational interaction becomes dominant at the length scale of $\mathcal{O}(g_{D3}^{1/4}\sqrt{\alpha'})$ [5]. Thus, action, (2), which does not include the gravitational interaction, should apply, and consequently the dyon solutions derived should be accurate. On the other hand, let $g_{D3} \rightarrow 1/g_{D3}$ so $L_{D3} \gg 1$; then the D-string, also a BPS state, becomes lighter than the F-string. As a consequence of weak/strong duality, the dyon solutions should still be applicable with the F-strings being replaced by D-strings and the dyon electric and magnetic charges being interchanged.

After applying a T-duality transformation to the dyon solutions obtained in Section 2.1, we have obtained dyon solutions residing on $D3$-branes where the effect of the gravitational interaction is apparent. This occurs despite the fact that action, (2), does not explicitly include the gravitational interaction. The presence of gravitational effects in this case is an example of how, in string theory, one-loop open string interactions, that is, Yang-Mills interactions, are related to tree level, closed string interactions, that is, gravitational interactions. In Figure 4, we depict, for illustrative purposes, two parallel Dp-branes in close proximity. The two Dp-branes can interact through open strings which connect the two Dp-branes. In the figure, we show the one-loop vacuum graph for such an interaction which can be interpreted as an open string moving in a loop. Alternatively, the interaction can be interpreted as a closed string being exchanged between two Dp-branes. In a certain sense, spin-2 gravitons, that is, the closed strings in their massless state, comprise a bound state of spin-one Yang-Mills bosons, that is, open strings in their massless state.

Prior to the application of the T-duality transformation, the open strings can propagate anywhere in the $D4$-brane.

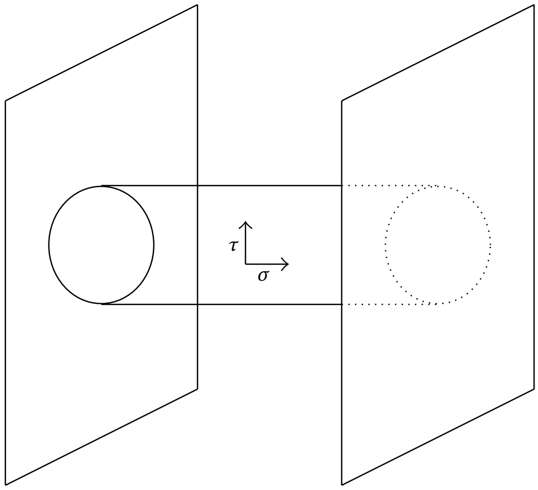

Figure 4: One-Loop String Diagram. Shown is the diagram for an open string, whose end points are fixed on two different Dp-branes, propagating in a loop, τ being the time variable on the world sheet. Alternatively, interchanging τ with σ, the same diagram can be interpreted as a closed string being exchanged between the two Dp-branes.

After the T-duality transformation, the open strings are constrained to propagate only within $D3$-branes, whereas the closed strings can still propagate in the bulk region between the $D3$-branes; that is, gravitons can propagate in spacetime dimensions not allowed for the Yang-Mills bosons. Based on certain general assumptions, Weinberg and Witten have shown the impossibility of constructing a spin-2 graviton as a bound state of spin-1 gauge fields [12]. One of the assumptions on which the proof is based is that the spin-1 gauge bosons and spin-2 gravitons propagate in the same spacetime dimensions. In the example presented here, this assumption is violated so that the conclusion of their theorem is avoided. These dyon solutions, thus, provide a simple example of gauge/gravity duality, which is discussed in detail in the work of Polchinski [13].

3. Conclusions

We have investigated dyon solutions within the context of superstring theory. Beginning with type I $SO(32)$ superstring theory in ten dimensions, six of the spatial dimensions of which are compact, we have applied the group of T-duality transformations to five of the compact dimensions. The result is 16 $D4$-branes, a number n ($2 \leq n \leq 16$) of which are coincident. The five T-dualized dimensions, whose size is taken to be vanishingly small, become the five internal spacetime dimensions while the remaining five dimensions correspond to the external 4 + 1-dimensional spacetime. Making a suitable ansatz for the gauge fields residing on the n coincident $D4$-branes, we have obtained dyon solutions from an action consisting of two terms: the 4 + 1-dimensional, non-Abelian Dirac-Born-Infeld action and a Wess-Zumino-like action. The former action gives the low energy effective coupling of $D4$-branes to NS-NS closed strings and the latter of $D4$-branes to R-R closed strings. The method of solution involves transforming the 4 + 1-dimensional action from the Lagrangian formalism to the Hamiltonian formalism and then seeking solutions which minimize the energy. The resulting dyon solutions, which are BPS states, reside on the n $D4$-branes and are therefore associated with a supersymmetric $U(n)$ gauge theory in 4 + 1 spacetime dimensions. These dyon solutions can be alternatively understood in the limit when the size of the remaining compact spacetime dimension, x^4, approaches zero. In this situation, the 4 + 1-dimensional spacetime is reduced to a 3 + 1-dimensional spacetime. As a consequence, the A_4 component of the vector potential becomes a Lorentz scalar with respect to 3 + 1-dimensional spacetime and can be interpreted as a Higgs boson transforming as the adjoint representation of the $U(n)$ gauge group, analogous to the Higgs boson associated with the 't Hooft/Polyakov magnetic monopole. Finally, we perform a T-duality transformation in the x^4-direction. As a result, $n-2$ of the $D4$-branes are transformed into $n-2$ coincident $D3$-branes, whose intrinsic geometry is flat. The remaining two $D4$-branes are transformed into two separate $D3$-branes whose intrinsic geometry is curved. As depicted in Figure 3, the two $D3$-branes are joined by wormhole at $r = 0$. The scalar curvature of each $D3$-brane reaches a maximum, finite value, at $r = 0$ and approaches zero as $r \rightarrow \infty$. The dyon resides on these two $D3$-branes. Furthermore, the values of electric and magnetic charges of the dyon on one $D3$-brane are minus the values on the other $D3$-brane, and as a consequence the electric and magnetic field lines enter the wormhole from one $D3$-brane and exit from the other $D3$-brane.

The T-duality transformation in the x^4-direction causes two of the $D3$-branes to acquire intrinsic curvature. This occurs despite the fact that the Lagrangian density from which the dyon solutions have been obtained does not explicitly include the gravitational interaction. This can be understood heuristically from the open string, one-loop vacuum graph given in Figure 4. From one perspective, the graph describes an open string, whose ends are fixed on two different $D3$-branes, moving in a loop, or, alternatively, the exchange of a closed string between two $D3$-branes. Thus, the gravitational interaction, that is, the closed string interaction, and the Yang-Mills interaction, that is, the open string interaction, appear as alternative descriptions of the same interaction. This simple example is suggestive of the subtle, but profound, connection between the Yang-Mills and gravitational interactions, specifically gauge/gravity duality.

Appendices

A. Units and Conventions

Concerning conventions, the Minkowski signature is $(- + + + \ldots)$, and the Levi-Cività symbols $\epsilon_{0123} = \epsilon_{123} = 1$. Other relevant conventions are as follows: Greek letters denote four-dimensional spacetime indices, that is, 0, 1, 2, and 3, whereas capitalized Roman letters are used when the spacetime dimension is greater than four. The small Roman letters, i, j, k, l, and m, are reserved for spatial dimensions in four-dimensional spacetime, that is, 1, 2, and 3. The small Roman letters, a, b, c, d, are used to enumerate the generators of the Lie group. The Levi-Cività tensor in three space dimensions is $\varepsilon_{ijk} = \sqrt{|\det(\widetilde{G}_{ij})|}\epsilon_{ijk}$, where $\widetilde{G}_{ij}$ are the spatial components of the metric tensor. We focus our attention on Yang-Mills theories based on the compact Lie groups, $U(n)$. A typical group element u ($u \in U(n)$) is represented in terms of the group parameters θ_a ($a = 0 \cdots n^2 - 1$) as $u = e^{i\theta_a T^a}$. The generators of this group in the fundamental representation are denoted by T^a ($a = 0 \cdots n^2 - 1$). The generator T^0 generates the $U(1)$ portion of $U(n)$, and the remaining T^a generate the $SU(n)$ portion. The Lie algebra of the group generators, T^a, is $[T^b, T^c] = if^{abc}T^c$, with f^{abc} being the structure constants of $U(n)$. The generators of $U(n)$ are required to satisfy the trace condition $\mathrm{Tr}(T^aT^b) = \delta^{ab}/2$. Thus, in particular, $T^0 = (1/\sqrt{2n})I_n$, I_n being the n-dimensional identity matrix. The Yang-Mills coupling constant is denoted by g^2_{D3}. We employ Lorentz-Heaviside units of electromagnetism so that $c = \hbar = \epsilon_0 = \mu_0 = 1$; consequently, the Dirac quantization condition is $g_{D3}g_m = 2\pi$. The quantity g_m is the magnetic charge of a unit charged Dirac monopole.

Consistent with our analysis, the Yang-Mills-Higgs Lagrangian is

$$\begin{aligned}\mathscr{L} &= -\frac{1}{2g^2_{D3}}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \mathrm{Tr}\left(\mathscr{D}_\mu W \mathscr{D}^\mu W\right) \\ &\quad - V\left(2\,\mathrm{Tr}\left[WW\right]\right) \\ &= -\frac{1}{4g^2_{D3}}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}\mathscr{D}_\mu W^a \mathscr{D}^\mu W^a \\ &\quad - V\left(W^aW^a\right),\end{aligned} \tag{A.1}$$

where V is a potential function depending on the Higgs field, W, and

$$\begin{aligned}F_{\mu\nu} &= F^a_{\mu\nu}T^a, \\ W &= W^aT^a, \\ A_\mu &= A^a_\mu T^a.\end{aligned} \tag{A.2}$$

The covariant derivative, $\mathscr{D}_\mu$, is defined as

$$\begin{aligned}\mathscr{D}_\mu &\equiv \partial_\mu - iA_\mu, \\ F_{\mu\nu} &= -i\left[\mathscr{D}_\mu, \mathscr{D}_\nu\right].\end{aligned} \tag{A.3}$$

Thus,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu. \tag{A.4}$$

The Higgs field W is a scalar transforming as the adjoint representation of $U(n)$ so that its covariant derivative is

$$\mathscr{D}_\mu W = \partial_\mu W - i\left[A_\mu, W\right] = \partial_\mu W^a + f^{abc}A^b_\mu W^c. \tag{A.5}$$

B. Relationships among the Various String Theory Parameters

In principle, string theory has no adjustable parameters other than its characteristic length scale, $\sqrt{\alpha'}$; however, various parameters of the theory do depend on values of the background fields. For reference, we provide explicit relationships between various string theory parameters and α'. Let Φ_0 be the vacuum expectation value of the dilaton background; that is, $\phi_0 = \langle\Phi\rangle$, Φ being the dilaton background. The closed string coupling constant, g, is

$$g = e^{\Phi_0}. \tag{B.1}$$

The physical gravitational coupling, κ, is

$$\kappa^2 \equiv \kappa^2_{10}g^2 = 8\pi G_N = \frac{1}{2}(2\pi)^7\alpha'^4 g^2, \tag{B.2}$$

where G_N is Newton's gravitational constant in 10 dimensions and $\kappa^2_{10} = \kappa^2_{11}/2\pi R$. The quantity κ_{11} is the gravitational constant appearing in the eleven-dimensional, low energy effective action of supergravity, and R is the compactification radius for reducing the eleven-dimensional theory to ten dimensions. The physical Dp-brane tension, τ_p, is

$$\begin{aligned}\tau_p \equiv \frac{T_p}{g} &= \frac{1}{g(2\pi)^p\alpha'^{(p+1)/2}} \\ &= \left(2\kappa^2\right)^{-1/2}(2\pi)^{(7-2p)/2}\alpha'^4,\end{aligned} \tag{B.3}$$

where T_p is Dp-brane tension. The Dp-brane R-R charge, μ_p, is

$$\mu_p = g\tau_p = \frac{1}{(2\pi)^p\alpha'^{(p+1)/2}}. \tag{B.4}$$

The coupling constant g_{Dp} of the $U(n)$ Yang-Mills theory on a Dp-brane is given by

$$g^2_{Dp} = \frac{1}{(2\pi\alpha')^2\tau_p} = (2\pi)^{p-2}g\alpha'^{(p-3)/2}. \tag{B.5}$$

The ratio of the F-string tension, τ_{F1}, to the D-string ($D1$-brane) tension, τ_{D1}, is

$$\frac{\tau_{F1}}{\tau_{D1}} = g. \tag{B.6}$$

C. Evaluation of the Dirac-Born-Infeld Determinant in an Arbitrary Number of Dimensions

In this appendix, we provide a heuristic derivation of the formula for evaluating an N'-dimensional ($N' = p + 1$) determinant of the form $\det(\widetilde{G}_{AB} + 2\pi\alpha' F_{AB})$ ($A, B = 0 \cdots p$), (2) and (5). (The notation used in this appendix does not adhere strictly to the font conventions defined in Appendix A.) Without loss of generality, we assume that the metric is diagonal. Consequently, we can express the determinant in the generic form

$$\det(h)\,\epsilon_{IJ\cdots OP} = \epsilon_{ij\cdots op} h_{iI} h_{jJ} \cdots h_{oO} h_{pP}, \tag{C.1}$$

where $h_{ij} = f_{ij} = 2\pi\alpha' F_{(i-1)(j-1)}$, $(i, j = 1 \cdots N')$ and $h_{(i)i} = g_{(i)i} = \widetilde{G}_{(i-1)(i-1)}$. Note: indices enclosed within parentheses are not summed.

Thus, the right-hand side of (C.1) comprises a sum of terms, each of which consists of products of the metric, $g_{(i)i}$ or f_{ij}. We need only to consider terms in which the number of f_{ij} in each product is even, since terms containing products of an odd number of f_{ij} vanish because $F_{AB} = -F_{BA}$. Thus, $\det(h_{ij})$, (C.1), can be reexpressed as a sum

$$\begin{aligned}&\det(h)\,\epsilon_{IJ\cdots OP}\\&\quad = \left(H_0 + H_2 + H_4 + H_{2n'} + \cdots\right)\epsilon_{IJ\cdots OP}.\end{aligned} \tag{C.2}$$

Each $H_{2n'}$ is a term in (C.1) which contains a product of f_{ij} which is even in number. The values of $2n'$ range from 0 to N' or $N'-1$ depending on whether N' is even or odd. The value of H_0, which contains no off-diagonal elements, f_{ij}, is evaluated as

$$H_0 = \det(g) \equiv g_{11} g_{22} \cdots g_{N'N'}. \tag{C.3}$$

In order to understand the structure of $H_{2n'}$, for arbitrary n', we first study the structure of H_4:

$$\begin{aligned}H_4 \epsilon_{IJ\cdots OP} &= \epsilon_{IJ\cdots klmn\cdots OP}\, g_{(I)I} g_{(J)J}\\&\quad\cdots g_{(O)O} g_{(P)P} f_{kK} f_{lL} f_{mM} f_{nN}.\end{aligned} \tag{C.4}$$

Equation (C.4) represents a term in (C.1) where the values of (k, l, m, n) in the sum are restricted to the specific integer values (K, L, M, N) from the set of integers $(1, 2 \cdots N')$. Multiplying (C.4) by the Levi-Cività symbol (which does not change value of the expression), we obtain

$$\begin{aligned}H_4 &= g_{(I)I} g_{(J)J}\\&\quad\cdots g_{(O)O} g_{(P)P} \epsilon_{IJ\cdots klmn\cdots OP} f_{kK} f_{lL} f_{mM} f_{nN} \epsilon_{IJ\cdots KLMN\cdots OP}.\end{aligned} \tag{C.5}$$

We now rearrange the terms in (C.5) in a form which is more useful for the subsequent analysis. By virtue of the Levi-Cività symbols, none of the (k, l, m, n) is equal to any of the others, and similarly for (K, L, M, N); however, each of the (k, l, m, n) is equal to one of the (K, L, M, N). Because of the antisymmetry of f_{ij}, some of the terms in the sum vanish, that is, whenever $k = K$, or whenever $l = L$, and so forth. By explicit construction or by a combinatorics argument, we can show that there are nine terms which are nonvanishing. We consider one typical term in the sum, for example, the term, $\{1\}$,

$$\{1\} = \{m = K, k = L, n = M, l = N\}. \tag{C.6}$$

We now show how to reexpress (C.5) so half, that is, 2, of the four f_{ij} are associated with one of the Levi-Cività symbols and the other half are associated with the other Levi-Cività symbol. In order for the two f_{ij} to be associated with one Levi-Cività symbol, the four subscripts on the two f_{ij} must be different. We associate the first f_{kK} with the Levi-Cività symbol to its left. To determine the remaining associations, we proceed as follows. Since $K = m$, we associate f_{mM} with the Levi-Cività symbol to the right. We now consider the second term in (C.5), that is, f_{lL}. Since $L = k$, we associate f_{lL} with the Levi-Cività symbol to its right and f_{nN} with the Levi-Cività symbol to the right. We continue in this manner to the next remaining term, f_{nN}. Since N does not equal any of the lowercase Roman subscripts associated with the Levi-Cività symbol to the left ($N = l$), we assign f_{nN} to the Levi-Cività symbol to the left. This completes the process since each f_{ij} is associated with either of the two Levi-Cività symbols. Using (C.6) we reexpress the subscripts on the Levi-Cività symbols in (C.5):

$$\begin{aligned}H_{4,\{1\}} &= g_{(I)I} g_{(J)J}\\&\quad\cdots g_{(O)O} g_{(P)P} \epsilon_{IJ\cdots(k)NK(n)\cdots OP} f_{kK} f_{nN} f_{mM} f_{lL} \epsilon_{IJ\cdots(m)LM(l)\cdots OP}.\end{aligned} \tag{C.7}$$

Now, we permute the subscripts of the Levi-Cività symbols so that the corresponding lowercase and uppercase Roman letters are adjoining. Both uppercase M and K require the same number of movements as the uppercase N and L do. Since the number of permutations is even, no change in sign of the Levi-Cività symbols results from permuting the subscripts. Equation (C.7) becomes

$$\begin{aligned}H_{4,\{1\}} &= g_{(I)I} g_{(J)J}\\&\quad\cdots g_{(O)O} g_{(P)P} \epsilon_{IJ\cdots(k)(K)(n)(N)\cdots OP} f_{kK} f_{nN} f_{mM} f_{lL} \epsilon_{IJ\cdots(m)(M)(l)(L)\cdots OP}.\end{aligned} \tag{C.8}$$

Each of the remaining $H_{4,\{s\}}$ $(s = 2 \cdots 9)$ can be expressed, similarly.

In order to understand, in generic terms, the structure of H_4, consider the following expression H'_4:

$$\begin{aligned}H'_4 &= g_{(I)I} g_{(J)J}\\&\quad\cdots g_{(O)O} g_{(P)P} \epsilon_{IJ\cdots kKnN\cdots OP} f_{kK} f_{nN} f_{mM} f_{lL} \epsilon_{IJ\cdots mMlL\cdots OP}.\end{aligned} \tag{C.9}$$

The expression H'_4 differs from the individual term $H_{4,\{1\}}$ in that the repeated indices (k, K, n, N) and (m, M, l, L) are summed. By inspection, the term $H_{4,\{1\}}$, as well as the remaining terms $H_{4,\{s\}}$ $(s = 2 \cdots 9)$, is contained in H'_4. Overtly, the number of nonvanishing terms in (C.9) is $(4!)^2$, each factor of $(4!)$ coming from each one of the Levi-Cività symbols. By a combinatorics argument, each factor of 4! is eightfold redundant so that the number of independent terms associated with each Levi-Cività symbol is three. Consequently, the number of independent terms in H'_4 is 9, that is, 3×3, with redundancy 8×8. Thus,

$$H_4 = \frac{1}{8 \times 8} H'_4. \tag{C.10}$$

Using similar reasoning, we can show that

$$H_{2n'} = \frac{H'_{2n'}}{R'_{2n'} \times R'_{2n'}}, \tag{C.11}$$

where

$$R'_{2n'} = 2^{n'} n'!. \tag{C.12}$$

In order to show (C.11), one needs to use the fact that the number of nonvanishing terms, $R_{2n'}$, in $H_{2n'}$ is given by

$$\begin{aligned} R_{2n'} &= \left(2n'-1\right) R_{2(n'-1)} \left(2n'-1\right) \\ &= \left[\left(2n'-1\right)!!\right]^2, \end{aligned} \tag{C.13}$$

so that the number of independent terms, $r_{2n'}$, associated with each of the two Levi-Cività symbols of $H'_{2n'}$ is

$$r_{2n'} = \left(2n'-1\right)!!. \tag{C.14}$$

Thus,

$$R'_{2n'} = \frac{\left(2n'\right)!}{r_{2n'}} = 2^{n'} n'!. \tag{C.15}$$

Both (C.13) and (C.14) are obtained from combinatorics arguments. Using the metric tensor to lower indices in f^{ij} and the relationship (C.15), we reexpress (C.11)

$$\begin{aligned} H_{2n'} &= -\det\left(g\right) \frac{1}{\left(N'-2n'\right)!} \left(2n'-1\right)!!^2 \\ &\times {}^*\left(\underbrace{f \wedge f \wedge \cdots f}_{n'}\right) \\ &\cdot {}^*\left(\underbrace{f \wedge f \wedge \cdots \wedge f}_{n'}\right). \end{aligned} \tag{C.16}$$

The Hodge $*$ operation transforms an s-form Q in an N'-dimensional space to an $(N'-s)$-form whose components are

$$\begin{aligned} \left({}^*Q\right)_{i_{s+1}\cdots i_{N'}} &= Q^{i_{j_1}\cdots j_s} \frac{1}{s!} \sqrt{\det\left|g\right|} \epsilon_{j_1 j_2 \cdots j_s i_{s+1} \cdots i_{N'}}, \\ {}^*Q \cdot {}^*Q &\equiv \left({}^*Q\right)_{i_{s+1}\cdots i_{N'}} \left({}^*Q\right)^{i_{s+1}\cdots i_{N'}}. \end{aligned} \tag{C.17}$$

Note that in (C.16) $\underbrace{f \wedge f \wedge \cdots f}_{n'}$ is a $2n'$-form. Also, in (C.16), the minus sign to the right of the equal sign is a consequence of the Minkowski signature of the metric; that is, $\epsilon^{12\cdots N'} = -\epsilon_{12\cdots N'}$. For Euclidean signature, the minus sign is replaced by a plus sign. Using properties of the Levi-Cività symbol, we can show that

$$H_2 = \det\left(g\right) \frac{1}{2!} f_{ij} f^{ij}, \tag{C.18}$$

irrespective of the signature of the metric. Using (C.2), (C.16), and (C.18), we obtain

$$\begin{aligned} \det\left(h\right) = \det\left(g\right) \Bigg[& 1 + \frac{1}{2!} f_{ij} f^{ij} \\ & \mp \sum_{2n'=4}^{2n'=N''} \Bigg(\frac{1}{\left(N'-2n'\right)!} \\ & \times \left(2n'-1\right)!!^{2} {}^*\left(\underbrace{f \wedge f \wedge \cdots f}_{n'}\right) \\ & \cdot {}^*\left(\underbrace{f \wedge f \wedge \cdots \wedge f}_{n'}\right) \Bigg) \Bigg], \end{aligned} \tag{C.19}$$

where

$$N'' = \begin{cases} N' & \text{if } N' \text{ is even} \\ N'-1 & \text{if } N' \text{ is odd.} \end{cases} \tag{C.20}$$

The minus (plus) sign corresponds to a metric with Minkowski (Euclidean) signature.

For the case when $N' = 5$ and the metric has Minkowski signature, (C.19) reduces to

$$\begin{aligned} \det\left(h\right) = \det\left(g\right) \Bigg[& 1 + \frac{1}{2!} f_{ij} f^{ij} - \frac{1}{\left(5-2\cdot 2\right)!} \\ & \times \left(3^2 \cdot 1^2\right) {}^*\left(f \wedge f\right)_k {}^*\left(f \wedge f\right)^k \Bigg]. \end{aligned} \tag{C.21}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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